If G is a group, H is a subgroup of G, and K is a subgroup of H, then K is a subgroup of G.

If G is a group, and H and K are subgroups of G, then their intersection $H \cap K$ is a subgroup of G.

A permutation $\alpha \in S_n$ is called *even* if it can be written as a product of an even number of transpositions; otherwise it is called *odd*. Exactly half of the permutations in S_n are even.

Set

$$A_n = \{\alpha \in S_n \mid \alpha \text{ is even}\}.$$

Then A_n is a subgroup of S_n, called the *alternating subgroup*.

Let H be a subgroup of S_n. Then either H consists of even permutations or exactly half of the permutations in H are even. We prove this now.

Problem 1. Let $H \leq S_n$. Show that either $H \leq A_n$ or $|H| = 2|H \cap A_n|$.

Solution. If G is a group, $x \in G$, and $Y \subset G$, define

$$xY = \{xy \mid y \in Y\}.$$

Suppose that H is not contained in A_n, and let $\alpha \in H \smallsetminus A_n$. Let K denote the set of even permutation in H, and let L denote the set of odd permutations in H. Then $K \cup L = H$, and $K \cap L = \varnothing$. Thus $|H| = |K| + |L|$.

Since H is closed under composition, $\alpha K \subset H$ and $\alpha L \subset H$. Since the product of an odd and and even permutation is odd, and the product of two odd permutations is even, we see that actually $\alpha K \subset L$ and $\alpha L \subset K$. Consider the map

$$f_1 : K \to L \text{ given by } f_1(\kappa) = \alpha\kappa,$$

and the map

$$f_2 : L \to K \text{ given by } f_2(\lambda) = \alpha\lambda.$$

Since left multiplication by α^{-1} produces an inverse for these maps, it is clear that both are injective. Therefore there exists a bijective functions $K \to L$, showing that these two sets have the same cardinality. This shows that $|H| = 2|K|$. □

Let $\rho, \tau \in S_n$ be given by

$$\rho = (1 \quad 2 \quad \dots \quad n) \quad \text{and} \quad \tau = \begin{cases} (2 \quad n)(3 \quad n-1)\dots(\frac{n+1}{2} \quad \frac{n+3}{2}) & \text{if } n \text{ is odd;} \\ (2 \quad n)(3 \quad n-1)\dots(\frac{n}{2} \quad \frac{n}{2}+2) & \text{if } n \text{ is even.} \end{cases}$$

Set

$$D_n = \{\epsilon, \rho, \rho^2, \dots, \rho^{n-1}, \tau, \tau\rho, \tau\rho^2, \dots, \tau\rho^{n-1}\} \subset S_n.$$

Then D_n is a subgroup of S_n, called the *dihedral subgroup*. The proof that this is a subgroup follows from the identity $\tau\rho = \rho^{n-1}\tau$. Clearly, $|D_n| = 2n$.

Set $K_n = D_n \cap A_n$. Then K_n is a subgroup of S_n, and either $K_n = D_n$ or K_n is exactly half of D_n. Thus $|K_n| = 2n$, or $|K_n| = n$.

Problem 2. Let $n = 4$.

(a) Compute ρ and τ in this case.

(b) Show that K_4 is a noncyclic abelian subgroup of S_4.

Solution. We have

$$\rho = (1\ 2\ 3\ 4) \quad \text{and} \quad \tau = (2\ 4).$$

Also,

$$K_4 = \{\epsilon, (1\ 2)(3\ 4), (1\ 3)(2\ 4), (1\ 4)(2\ 3)\}.$$

Every element of K_4 has order two, so K_4 is not cyclic (of order four). Computation shows that the group is abelian. □

Problem 3. Let $n = 5$.

(a) Compute ρ and τ in this case.

(b) Show that $K_5 = D_5$.

Solution. We have

$$\rho = \texttt{(1 2 3 4 5)} \quad \text{and} \quad \tau = \texttt{(2 5)(3 4)}.$$

Since ρ is an even permutation, so are all of its powers. Moreover, all of the reflections fix exactly one point and transpose two pairs of points; thus they are even. Since each member of K_5 is even, we have $K_5 = D_5$. $\square$

Problem 4. Let $n = 7$.

(a) Compute ρ and τ in this case.

(b) Show that K_7 is a cyclic subgroup of S_7.

Solution. We have

$$\rho = \texttt{(1 2 3 4 5 6 7)} \quad \text{and} \quad \tau = \texttt{(2 7)(3 6)(4 5)}.$$

Since ρ is even, all of its powers are in K_7. Moreover, since n is odd, each of the reflections fixes exactly one point, and so is the product of three transpositions, and is therefore odd. So, $K_7 = \langle \rho \rangle$ is cyclic. $\square$

Problem 5. Try to generalize the previous problems: what can you say about K_n in the following cases?

(a) $n \equiv 0 \pmod 4$

(b) $n \equiv 1 \pmod 4$

(c) $n \equiv 2 \pmod 4$

(d) $n \equiv 3 \pmod 4$

Solution. We will discuss **(a)** and **(c)** together.

(a) and **(c)** Suppose n is even. Then ρ is an odd permutation, but ρ^2 is even, and exactly half of $\langle \rho \rangle$ is contained in A_n.

Also, τ fixes two points, so τ moves $(n/2) - 2$ pairs of points; and $\tau\rho$ does not fix any points (it is reflection through the midpoints of opposite edges), so $\tau\rho$ moves $n/2$ pairs of points.

If $n \equiv 0 \pmod 4$, then $n/2$, so $\tau\rho$ is even, and τ is odd, in which case $K_n = \langle \rho^2, \tau\rho \rangle$.

If $n \equiv 2 \pmod 4$, then $n/2 - 2$ is even, so τ is even, and $\tau\rho$ is odd, in which case $K_n = \langle \rho^2, \tau \rangle$.

In either case, exactly half of the reflections are in K_n. It turns that in either case, K_n is isomorphic to D_m, where $m = \dfrac{n}{2}$.

(b) Suppose that $n \equiv 1 \pmod 4$. Since n is odd, ρ is an even permutation. All reflections fix exactly one point, so they move $\dfrac{n-1}{2}$ pairs of points. Since $n \equiv 1 \pmod 4$, $\dfrac{n-1}{2}$ is even, so all of these reflections are in A_n. Thus, $D_n \subset A_n$, and $K_n = D_n$.

(d) Suppose that $n \equiv 3 \pmod 4$. Since n is odd, ρ is an even permutation. All reflections fix exactly one point, so they move $\dfrac{n-1}{2}$ pairs of points. Since $n \equiv 3 \pmod 4$, $\dfrac{n-1}{2}$ is odd, so none of these reflections are in A_n. Thus, $K_n = \langle \rho \rangle$ is cyclic. $\square$

Let n be a positive integer, and let $\mathbb{N}_n = \{k \in \mathbb{Z} \mid 1 \leq k \leq n\}$.

Problem 6. Let $n \in \mathbb{Z}$ with $n \geq 4$ and $n \equiv 2 \pmod 4$. Let $m = \dfrac{n}{2}$. Find an explicit isomorphism $\Phi : K_n \to D_m$.

Solution. In this case, the action of K_n on $\mathbb{N}_n$ is identical to the action of D_m on $\mathbb{N}_m$.

To picture this, imagine a regular hexagon, with an equilateral triangle inscribed inside, touching the odd vertices. Then ρ^2 will rotate this triangle, and the even reflections will transpose odd vertices. We can explicitly give the details.

Define $f : \mathbb{N}_n \to \mathbb{N}_m$ by $f(i) = \dfrac{i+1}{2}$. For each $\alpha \in K_n$, define $\phi_\alpha : \mathbb{N}_m \to \mathbb{N}_m$ by $\phi_\alpha(j) = f(\alpha(f^{-1}(j)))$. Define $\Phi : K_n \to D_m$ by $\Phi(\alpha) = \phi_\alpha$. Then Φ is an isomorphism. □

Problem 7. Let $n \in \mathbb{Z}$ with $n \geq 4$ and $n \equiv 0 \pmod 4$. Let $m = \dfrac{n}{2}$. Find an explicit isomorphism $\Phi : K_n \to D_m$.

Proof. Here, the even reflections to not preserve the parity of the vertex number. For example, in K_8, the reflections do not fix any points, and so they swap even and odd vertices. For example, `(1 8)(2 7)(3 6)(4 5)` is an even reflection.

However, we can view the action of K_8 on the *sides* of a regular octagon as identical to the action of D_4 on the *vertices* of a square. Visually, imagine a square inscribed in a regular octagon such that the vertices are the midpoints of alternating sides. Once this is imagined, we can explicitly write the function.

Let $X = \{\{i, i+1\} \mid i = 1, 3, \ldots, n-1\}$ be the collection of consecutive pairs of points in $\mathbb{N}_n$; view this as the set of sides of the regular n-gon. If $\alpha \in D_n$, and $x \in X$, then $\alpha(x) \in X$, since rigid motions of a regular n-gon will preserve its sides. Now K_n acts on X just as D_m acts on $\mathbb{N}_m$.

Let $f : X \to \mathbb{N}$ by $f(x) = (\max x)/2$. This is bijective. For $\alpha \in K_n$, define $\phi_\alpha : \mathbb{N}_m \to \mathbb{N}_m$ by $\phi_\alpha(j) = f(\alpha(f^{-1}(j)))$. Define $\Phi : K_n \to D_m$ by $\Phi(\alpha) = \phi_\alpha$. Then Φ is an isomorphism. □